On the Embeddedness of Minimal Surfaces

Aris Mercier

University of Warwick

2024

Measure Theory

The Problem of Measure

What would you say is the length of the interval $[1, 3]$?

The Problem of Measure

What would you say is the length of the interval $[1, 3]$?

$$\lambda([1, 3]) = 2.$$

The Problem of Measure

What would you say is the length of the interval $[1,3]$?

$$\lambda([1,3]) = 2.$$

How about $[-3,4] \cup [5,7]$?

The Problem of Measure

What would you say is the length of the interval $[1, 3]$?

$$\lambda([1, 3]) = 2.$$

How about $[-3, 4] \cup [5, 7]$?

$$\lambda([-3, 4] \cup [5, 7]) = 9.$$

The Problem of Measure

What would you say is the length of the interval $[1, 3]$?

$$\lambda([1, 3]) = 2.$$

How about $[-3, 4] \cup [5, 7]$?

$$\lambda([-3, 4] \cup [5, 7]) = 9.$$

If we are willing to work over the extended real line, we can reasonably say that

$$\lambda((-\infty, 2]) = +\infty.$$

The Problem of Measure

What would you say is the length of the interval $[1,3]$?

$$\lambda([1,3]) = 2.$$

How about $[-3,4] \cup [5,7]$?

$$\lambda([-3,4] \cup [5,7]) = 9.$$

If we are willing to work over the extended real line, we can reasonably say that

$$\lambda((-\infty,2]) = +\infty.$$

Finally, the "length" of a singleton set $\{a\}$ is zero.

The Problem of Measure

Now suppose you were asked to find the “length” of $\mathbb{Q} \cap [0, 1]$.

The Problem of Measure

Now suppose you were asked to find the "length" of $\mathbb{Q} \cap [0, 1]$.

There are two plausible answers to this:

The Problem of Measure

Now suppose you were asked to find the "length" of $\mathbb{Q} \cap [0, 1]$.

There are two plausible answers to this:

(i) We declare the question as meaningless, and remain satisfied with the idea that certain subsets of $\mathbb{R}$ do not have a well-defined notion of length;

The Problem of Measure

Now suppose you were asked to find the "length" of $\mathbb{Q} \cap [0,1]$.

There are two plausible answers to this:

(i) We declare the question as meaningless, and remain satisfied with the idea that certain subsets of $\mathbb{R}$ do not have a well-defined notion of length;

(ii) We claim that $\lambda(\mathbb{Q} \cap [0,1]) = 0$.

The Problem of Measure

Intuitively, a measure on $\mathbb{R}^n$ should be a function which maps subsets of $\mathbb{R}^n$ to elements of $[0, +\infty]$.

The Problem of Measure

Intuitively, a measure on $\mathbb{R}^n$ should be a function which maps subsets of $\mathbb{R}^n$ to elements of $[0, +\infty]$.

Unfortunately, there is no way to consistently assign a measure to *every* subset of $\mathbb{R}^n$ without running into paradoxes.

The Problem of Measure

Intuitively, a measure on $\mathbb{R}^n$ should be a function which maps subsets of $\mathbb{R}^n$ to elements of $[0, +\infty]$.

Unfortunately, there is no way to consistently assign a measure to *every* subset of $\mathbb{R}^n$ without running into paradoxes.

Instead, we restrict our attention to certain "measurable" subsets. These will include virtually every subset we will ever care about, so we are happy to make this small sacrifice.

The Problem of Measure

Intuitively, a measure on $\mathbb{R}^n$ should be a function which maps subsets of $\mathbb{R}^n$ to elements of $[0, +\infty]$.

Unfortunately, there is no way to consistently assign a measure to *every* subset of $\mathbb{R}^n$ without running into paradoxes.

Instead, we restrict our attention to certain "measurable" subsets. These will include virtually every subset we will ever care about, so we are happy to make this small sacrifice.

Definition (σ-algebra)

Let X be a set. A σ-algebra $\mathcal{A}$ on X is a collection of subsets of X such that:

(i) $\varnothing \in \mathcal{A}$;

(ii) $\mathcal{A}$ is closed under complements;

(iii) $\mathcal{A}$ is closed under countable unions.

The Problem of Measure

Definition (Measure)

Let X be a set, and $\mathcal{A}$ a σ-algebra on X. A *measure* on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, +\infty]$ such that:

The Problem of Measure

Definition (Measure)

Let X be a set, and $\mathcal{A}$ a σ-algebra on X. A *measure* on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, +\infty]$ such that:

(i) $\mu(\varnothing) = 0$;

The Problem of Measure

Definition (Measure)

Let X be a set, and $\mathcal{A}$ a σ-algebra on X. A *measure* on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, +\infty]$ such that:

(i) $\mu(\varnothing) = 0$;

(ii) If $\{E_n\}_n$ is a countable collection of pairwise disjoint sets in $\mathcal{A}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

The Problem of Measure

Definition (Measure)

Let X be a set, and $\mathcal{A}$ a σ-algebra on X. A *measure* on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, +\infty]$ such that:

(i) $\mu(\varnothing) = 0$;

(ii) If $\{E_n\}_n$ is a countable collection of pairwise disjoint sets in $\mathcal{A}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

The sets in $\mathcal{A}$ are called μ-*measurable*.

The Lebesgue Measure

Let's return to $\mathbb{R}^n$. We wish to construct a measure that agrees with and extends our intuitive notion of "volume".

The Lebesgue Measure

Let's return to $\mathbb{R}^n$. We wish to construct a measure that agrees with and extends our intuitive notion of "volume".

Theorem

There is a unique measure on $\mathbb{R}^n$ which is translation-invariant, complete, and assigns measure 1 to the unit cube, $[0, 1]^n$.

The Lebesgue Measure

Let's return to $\mathbb{R}^n$. We wish to construct a measure that agrees with and extends our intuitive notion of "volume".

Theorem

There is a unique measure on $\mathbb{R}^n$ which is translation-invariant, complete, and assigns measure 1 to the unit cube, $[0,1]^n$.

We call this the *Lebesgue measure*, and denote it by $\mathcal{L}^n$.

The Lebesgue Measure

The Lebesgue measure works great when we wish to measure n-dimensional subsets of $\mathbb{R}^n$.

The Lebesgue Measure

The Lebesgue measure works great when we wish to measure n-dimensional subsets of $\mathbb{R}^n$.

However, we often wish to measure k-dimensional subsets of $\mathbb{R}^n$, where $k < n$. The measure $\mathcal{L}^n$ cannot tell them apart, so we need something better.

The Lebesgue Measure

The Lebesgue measure works great when we wish to measure n-dimensional subsets of $\mathbb{R}^n$.

However, we often wish to measure k-dimensional subsets of $\mathbb{R}^n$, where $k < n$. The measure $\mathcal{L}^n$ cannot tell them apart, so we need something better.

Unfortunately, there is no longer an obvious choice of measure which agrees with our intuitive notion of volume.

The Hausdorff Measure

We choose to work with the k-dimensional Hausdorff (outer) measure. We denote this by $\mathcal{H}^k$ and define it as follows:

$$\mathcal{H}^k(A) := \lim_{\delta \downarrow 0} \mathcal{H}^k_\delta(A), \qquad A \subset \mathbb{R}^n$$

where, for each $\delta > 0$, $\mathcal{H}^k_\delta$ is defined by taking $\mathcal{H}^k_\delta(\varnothing) := 0$, and

$$\mathcal{H}^k_\delta(A) := \inf \sum_{i=1}^{\infty} \omega_k \left(\frac{\text{diam } C_i}{2} \right)^k$$

for any non-empty $A \subset X$. The infimum is taken over all countable collections $C_1, C_2, \ldots$ of subsets of X such that diam $C_i < \delta$ and $A \subset \bigcup_{i=1}^{\infty} C_i$. If no such collection exists, the right-hand side is taken to be $+\infty$.

Geometric Measure Theory

Densities

Let X be a metric space, μ an outer measure on X, and A a subset of X. We define the n-dimensional density of μ at $x \in A$ by

$$\Theta^n(\mu, A, x) := \lim_{\rho \downarrow 0} \frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}.$$

Densities

Let X be a metric space, μ an outer measure on X, and A a subset of X. We define the n-dimensional density of μ at $x \in A$ by

$$\Theta^n(\mu, A, x) := \lim_{\rho \downarrow 0} \frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}.$$

One can also define densities of measures with respect to other measures. These appear extensively in various "differentiation theorems" in GMT, and beyond.

Densities

Let X be a metric space, μ an outer measure on X, and A a subset of X. We define the n-dimensional density of μ at $x \in A$ by

$$\Theta^n(\mu, A, x) := \lim_{\rho \downarrow 0} \frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}.$$

One can also define densities of measures with respect to other measures. These appear extensively in various "differentiation theorems" in GMT, and beyond.

Under certain assumptions on X, one can prove that the density ratio

$$\frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}$$

is an increasing function of ρ.

Area Functional

For an open set $\Omega \subset \mathbb{R}^2$ and a function $u : \Omega \to \mathbb{R}$, the *graph* of u is the set

$$\mathcal{G}_u := \left\{ \left(x, u(x)\right) \in \mathbb{R}^3 : x \in \Omega \right\}.$$

Area Functional

For an open set $\Omega \subset \mathbb{R}^2$ and a function $u : \Omega \to \mathbb{R}$, the *graph* of u is the set

$$\mathcal{G}_u := \{ (x, u(x)) \in \mathbb{R}^3 : x \in \Omega \}.$$

We wish to impose some condition on u which ensures that the "area" of $\mathcal{G}_u$ is well-defined. One possibility is to require that u is locally Lipschitz on Ω.

Area Functional

For an open set $\Omega \subset \mathbb{R}^2$ and a function $u : \Omega \to \mathbb{R}$, the *graph* of u is the set

$$\mathcal{G}_u := \{ (x, u(x)) \in \mathbb{R}^3 : x \in \Omega \}.$$

We wish to impose some condition on u which ensures that the "area" of $\mathcal{G}_u$ is well-defined. One possibility is to require that u is locally Lipschitz on Ω.

The *area functional*, by definition, assigns to this graph its area. We have that

$$\mathcal{A}[u] := \int_\Omega \sqrt{1 + |\nabla u|^2}.$$

Area Functional

For an open set $\Omega \subset \mathbb{R}^2$ and a function $u : \Omega \to \mathbb{R}$, the *graph* of u is the set

$$\mathcal{G}_u := \left\{ \left(x, u(x)\right) \in \mathbb{R}^3 : x \in \Omega \right\}.$$

We wish to impose some condition on u which ensures that the "area" of $\mathcal{G}_u$ is well-defined. One possibility is to require that u is locally Lipschitz on Ω.

The *area functional*, by definition, assigns to this graph its area. We have that

$$\mathcal{A}[u] := \int_\Omega \sqrt{1 + |\nabla u|^2}.$$

Much like with functions on $\mathbb{R}$, we can ask what the critical points of this functional are.

Minimal Surfaces

A *surface* S is a 2-dimensional (topological) manifold. In particular, we do not assume that S is smooth.

Minimal Surfaces

A *surface* S is a 2-dimensional (topological) manifold. In particular, we do not assume that S is smooth.

A *minimal* surface is a critical point of the area functional.

Minimal Surfaces

A *surface* S is a 2-dimensional (topological) manifold. In particular, we do not assume that S is smooth.

A *minimal* surface is a critical point of the area functional. Note that a minimal surface does not actually need to minimise area.

Embeddedness

A minimal surface will, in general, contain branch points. A natural next step therefore, is to find sufficient conditions which guarantee that a minimal surface is smoothly embedded.

Embeddedness

A minimal surface will, in general, contain branch points. A natural next step therefore, is to find sufficient conditions which guarantee that a minimal surface is smoothly embedded.

Meeks and Yau proved that if $\Gamma = \partial M$ lies on the boundary of a convex set, then the minimal disc spanning Γ must be smoothly embedded.

Embeddedness

A minimal surface will, in general, contain branch points. A natural next step therefore, is to find sufficient conditions which guarantee that a minimal surface is smoothly embedded.

Meeks and Yau proved that if $\Gamma = \partial M$ lies on the boundary of a convex set, then the minimal disc spanning Γ must be smoothly embedded.

We now show that Γ having total curvature at most 4π is another such sufficient condition. For this we will need the following key result:

Extended Monotonicity Theorem

Theorem

Suppose that M is a compact 2-dimensional minimal submanifold of $\mathbb{R}^n$, $n > 2$, with rectifiable boundary $\Gamma := \partial M$. Consider a point $p \in \mathbb{R}^n$, and let $E = E(\Gamma, p)$ denote the exterior cone with vertex p over Γ:

$$E := \bigcup_{q \in \Gamma} \{tq + (1 - t)p : t \geqslant 1\}.$$

Extended Monotonicity Theorem

Theorem

Suppose that M is a compact 2-dimensional minimal submanifold of $\mathbb{R}^n$, $n > 2$, with rectifiable boundary $\Gamma := \partial M$. Consider a point $p \in \mathbb{R}^n$, and let $E = E(\Gamma, p)$ denote the exterior cone with vertex p over Γ:

$$E := \bigcup_{q \in \Gamma} \{tq + (1-t)p : t \geqslant 1\}.$$

Let $M' = M \cup E$. Then the density ratio

$$\frac{\mathcal{H}^2(M' \cap B(p,r))}{\pi r^2}$$

is an increasing function of r, for all $r > 0$.

Extended Monotonicity Theorem

Theorem

Suppose that M is a compact 2-dimensional minimal submanifold of $\mathbb{R}^n$, $n > 2$, with rectifiable boundary $\Gamma := \partial M$. Consider a point $p \in \mathbb{R}^n$, and let $E = E(\Gamma, p)$ denote the exterior cone with vertex p over Γ:

$$E := \bigcup_{q \in \Gamma} \{tq + (1-t)p : t \geqslant 1\}.$$

Let $M' = M \cup E$. Then the density ratio

$$\frac{\mathcal{H}^2(M' \cap B(p,r))}{\pi r^2}$$

is an increasing function of r, for all $r > 0$. That is,

$$\frac{d}{dr}\left(\frac{\mathcal{H}^2(M' \cap B(p,r))}{\pi r^2}\right) \geqslant 0,$$

with equality if and only if M' is a cone.

Interior Regularity

We are now ready to show that the interior of a minimal surface is embedded and free of branch points.

Interior Regularity

We are now ready to show that the interior of a minimal surface is embedded and free of branch points.

This is a proof by contradiction, and relies on the following three facts:

Interior Regularity

We are now ready to show that the interior of a minimal surface is embedded and free of branch points.

This is a proof by contradiction, and relies on the following three facts:

1. The density of M at an interior point p is bounded above by the density at p of the cone subtended by ∂M;

Interior Regularity

We are now ready to show that the interior of a minimal surface is embedded and free of branch points.

This is a proof by contradiction, and relies on the following three facts:

1. The density of M at an interior point p is bounded above by the density at p of the cone subtended by ∂M;

2. The density at p of this cone is at most $1/2\pi$ times the total curvature of ∂M;

Interior Regularity

We are now ready to show that the interior of a minimal surface is embedded and free of branch points.

This is a proof by contradiction, and relies on the following three facts:

1. The density of M at an interior point p is bounded above by the density at p of the cone subtended by ∂M;
2. The density at p of this cone is at most $1/2\pi$ times the total curvature of ∂M;
3. The density of M at any interior branch point is at least 2.

Interior Regularity

We are now ready to show that the interior of a minimal surface is embedded and free of branch points.

This is a proof by contradiction, and relies on the following three facts:

1. The density of M at an interior point p is bounded above by the density at p of the cone subtended by ∂M;
2. The density at p of this cone is at most $1/2\pi$ times the total curvature of ∂M;
3. The density of M at any interior branch point is at least 2.

The first statement is a consequence of extended monotonicity. The second statement follows from the Gauß–Bonnet theorem.

The Fáry–Milnor Theorem

From the previous embeddedness theorem, the Fáry–Milnor theorem follows as a simple corrolary. This is a fundamental result linking the geometry and topology of a simple closed curve in $\mathbb{R}^3$. It was proven independently by István Fáry in 1949 and by John Milnor in 1950.

The Fáry–Milnor Theorem

From the previous embeddedness theorem, the Fáry–Milnor theorem follows as a simple corrolary. This is a fundamental result linking the geometry and topology of a simple closed curve in $\mathbb{R}^3$. It was proven independently by István Fáry in 1949 and by John Milnor in 1950.

Corollary (Fáry–Milnor)

Let Γ be a simple closed curve in $\mathbb{R}^3$ with total curvature at most 4π. Then Γ is unknotted.

Thank you.